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ON THE RANDOM FUNCTIONAL CENTRAL LIMIT THEOREMS WITH ALMOST SURE CONVERGENCE

BY

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Dedicated to the memory of Professor Kazimierz Urbanik

Abstract. In this paper we present functional random-sum central limit theorems with almost sure convergence for independent nonidentically distributed random variables. We consider the case where the summation random indices and partial sums are independent. In the past decade several authors have investigated the almost sure functional central limit theorems and related 'logarithmic' limit theorems for partial sums of independent random variables. We extend this theory to almost sure versions of the functional random-sum central limit theorems.

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1. INTRODUCTION

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables, defined on a probability space $(\Omega, \mathscr{A}, P)$, such that $EX_n = 0$ and $EX_n^2 = \sigma_n^2 < \infty$, $n \geqslant 1$. Let us put $S_0 = 0$, $B_0^2 = 0$, $S_n = X_1 + \ldots + X_n$, $B_n^2 = \sigma_1^2 + \ldots + \sigma_n^2 = ES_n^2$, $n \geqslant 1$. Let $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued random variables, defined on the same probability space $(\Omega, \mathscr{A}, P)$. Assume that for each $n \geqslant 1$ the random variable N_n is independent of the random variables X_n, $n \geqslant 1$, and put

$$S_{N_n} = X_1 + \ldots + X_{N_n}, \quad B_{N_n}^2 = \sigma_1^2 + \ldots + \sigma_{N_n}^2.$$

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Let us put

$$M(t) = \max\{k \geqslant 0\colon\ B_k^2 \leqslant t\}, \qquad M_n(t) = M(tB_n^2), \qquad t \geqslant 0,$$

$$m(t) = \min\{k \geqslant 0\colon\ t \leqslant B_k^2\}, \qquad m_n(t) = m(tB_n^2), \qquad t \geqslant 0.$$

Then $M_n(1) = n = m_n(1)$ and, for every $t > 0$,

$$B_{M(t)}^2 \leqslant t \leqslant B_{m(t)}^2 \leqslant B_{M(t)+1}^2 \leqslant B_{M(t)}^2 + \max_{1 \leqslant k \leqslant M(t)+1} \sigma_k^2. \tag{1.1}$$

Assume that, for every $\varepsilon > 0$,

$$\frac{1}{B_{N_n}^2} \sum_{k=1}^{N_n} \int_{|x| \geqslant \varepsilon B_{N_n}^2} x^2\, dP(X_k \leqslant x) \xrightarrow{P} 0 \quad \text{as } n \to \infty, \tag{1.2}$$

where $\xrightarrow{P}$ denotes the convergence in probability. The condition (1.2) is called the *random Lindeberg condition*. Let us observe that the convergence in probability in (1.2) can be replaced by the convergence in mean. Thus if (1.2) holds, then

$$E(\max\{\sigma_k^2\colon 1 \leqslant k \leqslant N_n\}/B_{N_n}^2) \to 0 \quad \text{as } n \to \infty. \tag{1.3}$$

The condition (1.3) is called the *random Feller's condition*. We also note that if (1.2) holds, then by (1.1) and (1.3), for every $t > 0$,

$$E\{B_{M(tB_{N_n}^2)}^2/B_{N_n}^2\} \to t \quad \text{as } n \to \infty. \tag{1.4}$$

We introduce the usual "broken line process" on $[0, 1]$:

$$Y_n(t) = S_{M_n(t)}/B_n + X_{M_n(t)+1}(tB_n^2 - B_{M_n(t)}^2)/(B_n \sigma_{M_n(t)+1}^2), \qquad t \in [0, 1]. \tag{1.5}$$

It is clear that $Y_n(t) = S_k/B_n$ whenever $t = B_k^2/B_n^2$, $0 \leqslant k \leqslant n$, and $Y_n(t)$ is the straight line joining $(B_k^2/B_n^2, S_k/B_n)$ and $(B_{k+1}^2/B_n^2, S_{k+1}/B_n)$ in the interval $[B_k^2/B_n^2, B_{k+1}^2/B_n^2]$, $k = 0, 1, \ldots, n-1$. Thus $Y_n(t)$, $t \in [0, 1]$, is continuous with probability one, so that there is a measure P_n on the space $(C[0, 1], \mathscr{C})$, according to which the stochastic process $\{Y_n(t), 0 \leqslant t \leqslant 1\}$ is distributed. Of course, here and in what follows $C[0, 1]$ denotes the space of real-valued, continuous functions on $[0, 1]$ and $\mathscr{C}$ means the σ-field of Borel sets generated by the open sets of uniform topology.

It is well known that if (1.2) holds, then by Theorem 1 of Rychlik and Szynal [21] we have

$$Y_{N_n} \Rightarrow W \quad \text{as } n \to \infty, \tag{1.6}$$

where W denotes the standard Wiener measure on $(C[0, 1], \mathscr{C})$ with a corresponding standard Wiener process $\{W(t), 0 \leqslant t \leqslant 1\}$, and $\Rightarrow$ means the weak convergence of measures on the space $(C[0, 1], \mathscr{C})$.

In this paper we present an almost sure version of this theorem. Namely, let $\delta(x)$ denote the probability measure which assigns its total mass to

$x \in C[0, 1]$. Then, for every $\omega \in \Omega$, $\{\delta(Y_n(\omega)), n \geq 1\}$ is a sequence of probability measures on the space $(C[0, 1], \mathscr{C})$ and the distribution P_n of Y_n is just the average of the random measure $\delta(Y_n(\omega))$ with respect to P, i.e., for every $A \in \mathscr{C}$,

$$P_n(A) = \int_{\Omega} \delta\big(Y_n(\omega)(A)\big)\, dP(\omega).$$

The same concerns the sequence of probability measures $\{\delta(Y_{N_n}), n \geq 1\}$.

We shall form 'time averages' with respect to a logarithmic scale rather than 'space averages' and prove almost sure (a.s.) convergence for the resulting random measures. To be precise, we present a sufficient condition under which

$$(\log_+ B_{N_n}^2)^{-1} \sum_{k=1}^{N_n} (\sigma_{k+1}^2/B_k^2)\, \delta(Y_k) \Rightarrow W, \quad \text{as } n \to \infty, \text{ for almost every } \omega \in \Omega, \tag{1.7}$$

where $\log_+ x = \log x$ if $x \geq e$, and $\log_+ x = 1$ if $x < e$. The limit relation (1.7) will be called an *almost sure version of the random functional central limit theorem*. This remarkable property of the logarithmic means has intensively been studied in recent years and many extensions and variants of (1.7) have been obtained in the case when $P(N_n = n) = 1$, $n \geq 1$. In this case, several papers presented sufficient conditions under which (1.7) holds; see, e.g., Brosamler [6], Schatte [22]–[24], Lacey and Philipp [15], Atlagh [1], Rodzik and Rychlik [19], Ibragimov [12], Ibragimov and Lifshits [13], [14], Major [16], [17], Berkes [2], Berkes and Csáki [3], Fazekas and Rychlik [9], Rychlik and Szuster [20], and the references given in these papers. On the other hand, the case with random indices N_n, $n \geq 1$, has not been considered as so far. In this paper we extend this theory and show that (1.7) also holds under some additional conditions concerning the sequence $\{N_n, n \geq 1\}$. Let us observe that if $N_n = n$ with probability one, for every $n \geq 1$, then the random Lindeberg condition (1.2) holds if and only if $\{X_n, n \geq 1\}$ satisfies the Lindeberg condition, i.e., for every $\varepsilon > 0$,

$$\lim_{n \to \infty} B_n^{-2} \sum_{k=1}^{n} EX_k^2 I(|X_k| \geq \varepsilon B_n) = 0. \tag{1.8}$$

On the other hand, if (1.8) holds, then by Prokhorov's theorem (Prokhorov [18], cf. also Billingsley [5], Section 10) we have

$$Y_n \Rightarrow W \quad \text{as } n \to \infty \tag{1.9}$$

and, for every $t > 0$,

$$\lim_{n \to \infty} (B_{M_n(t)}^2/B_n^2) = t. \tag{1.10}$$

Furthermore, if (1.8) holds,

$$N_n \xrightarrow{P} \infty \quad \text{as } n \to \infty, \tag{1.11}$$

and, for every $n \geqslant 1$, N_n is independent of $\{X_n, n \geqslant 1\}$, then (1.2) also holds. Thus (1.8) and (1.11) imply (1.6). On the other hand, strong laws of large numbers for randomly indexed sequences need stronger assumptions than (1.11), see e.g. Gut [11], Chapter I. Of course, in the almost sure central (functional) limit theorems the convergence is almost sure, therefore the random indices case has its own meaning. Actually, (1.7) can be viewed as a weighted strong law of large numbers or a Glivenko–Cantelli type theorem (cf., Csörgő and Horváth [7]).

The purpose of this paper is to prove the almost sure version of the random functional central limit theorems. The presented results generalize, to sequences with random indices, the main theorems presented in the above-mentioned papers. We extend the basic results of Fazekas and Rychlik [9], and Berkes and Csáki [3] to sequences of random elements with random indices. In the proofs we shall also follow some ideas of Berkes and Csáki [3].

2. RESULTS

Let $BL = BL(B)$ be the class of functions $f: B \to R$ with $\|f\|_{BL} = \|f\|_L + \|f\|_\infty < \infty$, where

$$\|f\|_L = \sup\{|f(x)-f(y)|/\rho(x, y):\ x, y \in B,\ x \neq y\}, \tag{2.1}$$

and $\|f\|_\infty = \sup\{|f(x)|:\ x \in B\}$.

Let (B, ρ) be a separable and complete metric space and let $\{\zeta_n, n \geqslant 1\}$ be a sequence of B-valued random elements, defined on a probability space $(\Omega, \mathscr{F}, P)$. Let μ_ζ denote the distribution of the random element ζ. Let $\log_+ x = \log x$ if $x \geqslant 1$ and 0 otherwise. We will also denote by $\Rightarrow$ the weak convergence of measures on the space (B, ρ).

We can now formulate our general results providing the almost sure versions of the random functional central limit theorem.

THEOREM 1. *Let $\{\zeta_n, n \geqslant 1\}$ be a sequence of B-valued random elements. Let $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued random variables such that, for every $n \geqslant 1$, N_n is independent of ζ_k, $k \geqslant 1$. Assume that, for each $n \geqslant 1$, there exist B-valued random elements $\zeta_{k,n}$, $1 \leqslant k < n$, such that $\zeta_{k,n}$ are independent of ζ_k and N_n for $k < n$ and*

$$E\{\rho(\zeta_{k,n}, \zeta_n) \wedge 1\} \leqslant C\{\log_+ \log_+ (c_n/c_k)\}^{-(1+\varepsilon)} \tag{2.2}$$

for some constants $C > 0$, $\varepsilon > 0$ and an increasing sequence of positive numbers $\{c_n, n \geqslant 1\}$ such that

$$c_n \to \infty, \quad c_{n+1}/c_n = 0(1) \quad \textit{as } n \to \infty. \tag{2.3}$$

Let $\{d_n, n \geqslant 1\}$ be a sequence such that

$$0 \leqslant d_n \leqslant \log(c_{n+1}/c_n),\ n \geqslant 1, \quad D_n = d_1 + \ldots + d_n \to \infty \textit{ as } n \to \infty, \tag{2.4}$$

and

$$E\left(\log (D_{N_n} \vee 4)\right)^{-(1+\varepsilon)} \to 0 \quad \textit{as } n \to \infty. \tag{2.5}$$

Then, for any probability distribution μ on the Borel σ-algebra of B, the following relations are equivalent:

$$D_{N_n}^{-1} \sum_{k=1}^{N_n} d_k \delta_{\zeta_k} \Rightarrow \mu, \quad \textit{as } n \to \infty, \ P\textit{-a.s.}, \tag{2.6}$$

$$D_{N_n}^{-1} \sum_{k=1}^{N_n} d_k \mu_{\zeta_k} \Rightarrow \mu, \quad \textit{as } n \to \infty, \ P\textit{-a.s.} \tag{2.7}$$

Remark 1. Theorem 1 remains valid if condition (2.2) is replaced by the following:

$$E\{\rho(\zeta_{k,n}, \zeta_n) \wedge 1\} \leqslant C(c_k/c_n)^{\beta}, \quad 1 \leqslant k < n, \ n \geqslant 1, \tag{2.8}$$

for some constants $C > 0$ and $\beta > 0$. Furthermore, if (2.8) holds, then in Theorem 1 we can also choose

$$d_k = \log(c_{k+1}/c_k) \exp\left((\log c_k)^{\alpha}\right), \quad k \geqslant 1,$$

for any constant $0 \leqslant \alpha < \frac{1}{2}$.

Remark 2. Let us observe that if (2.4) holds, and

$$N_n \to \infty, \quad \text{as } n \to \infty, \ P\text{-a.s.}, \tag{2.9}$$

$$D_n^{-1} \sum_{k=1}^{n} d_k \mu_{\zeta_k} \Rightarrow \mu, \quad \text{as } n \to \infty, \tag{2.10}$$

then (2.7) and, for every $\varepsilon > 0$, (2.5) also hold. Furthermore, in the special case, if $\mu_{\zeta_n} \Rightarrow \mu$ as $n \to \infty$, then (2.10) is a consequence of (2.4). On the other hand, (2.7) can be satisfied even if (2.10) does not hold. The importance of condition (2.10) is demonstrated in Berkes et al. [4]. We also note that if

$$N_n \xrightarrow{P} \infty \quad \text{as } n \to \infty, \tag{2.11}$$

then (2.5) is a consequence of (2.4).

THEOREM 2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with $EX_n = 0$ and $0 < EX_n^2 = \sigma_n^2 < \infty$, $n \geqslant 1$. Let $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued random variables such that, for every $n \geqslant 1$, N_n is independent of $\{X_n, n \geqslant 1\}$. If* (1.8) *holds, and*

$$N_n \to \infty \ P\textit{-a.s.}, \quad \textit{as } n \to \infty, \tag{2.12}$$

then, for every $0 \leqslant \alpha < \frac{1}{2}$,

$$D_{N_n}^{-1} \sum_{k=1}^{N_n} d_k \delta(Y_k) \Rightarrow W, \quad \textit{as } n \to \infty, \textit{ for almost every } \omega \in \Omega, \tag{2.13}$$

and

$$\lim_{n\to\infty} \sup_x \Big|D_{N_n}^{-1} \sum_{k=1}^{N_n} d_k I(S_k \leqslant xB_k) - \Phi(x)\Big| = 0 \ P\text{-a.s.}, \tag{2.14}$$

where

$$d_k = \log(B_{k+1}^2/B_k^2)\exp\big((\log B_k^2)^\alpha\big), \quad k \geqslant 1,$$

$D_{N_n} = d_1 + \ldots + d_{N_n}$ *and* $\Phi(x)$ *is the standard normal distribution function.*

Let us observe that if, in Theorem 2, $\alpha = 0$, then

$$d_k = \log(1+\sigma_{k+1}^2/B_k^2) \sim \sigma_{k+1}^2/B_k^2 \quad \text{as } k\to\infty,$$

and

$$D_{N_n} = \log B_{N_n+1} - \log\sigma_1^2 \sim \log B_{N_n}^2, \quad \text{as } n\to\infty, \ P\text{-a.s.}$$

Thus, under the assumptions of Theorem 2, (1.7) also holds, and

$$\lim_{n\to\infty} \sup_x \Big|(\log_+ B_{N_n}^2)^{-1} \sum_{k=1}^{N_n} (\sigma_{k+1}^2/B_k^2) I(S_k \leqslant xB_k) - \Phi(x)\Big| = 0 \ P\text{-a.s.} \tag{2.15}$$

Let us also note that (2.14) and (2.15) actually give strong versions of the random central limit theorem. Of course, (2.14) and (2.15) are consequences of Theorem 5.1 of Billingsley [5] and (2.13) or (1.7), respectively. Namely, if h is a measurable mapping from $C[0, 1]$ into another metric space S with metric ρ and σ-field $\mathscr{S}$ of Borel sets, then every probability measure P on $(C[0, 1], \mathscr{C})$ induces on $(S, \mathscr{S})$ the image measure Ph^{-1}, defined by $Ph^{-1}(A) = P\big(h^{-1}(A)\big)$ for $A \in \mathscr{S}$. Thus, by Theorem 2 and Theorem 5.1 of Billingsley [5], we get

$$D_{N_n}^{-1} \sum_{k=1}^{N_n} d_k \delta h^{-1}(Y_k) \Rightarrow Wh^{-1}, \quad \text{as } n\to\infty, \ P\text{-a.s.} \tag{2.16}$$

for every measurable $h\colon C[0, 1]\to S$ which is continuous W-a.e. Hence, setting $h(x) = x(1)$ we get (2.14) from (2.13), and (2.15) from (1.7), respectively. We may also obtain pointwise asymptotic results for the following functionals:

$$h_1(x) = \sup\{|x(t)|^p\colon 0\leqslant t\leqslant 1\},\ p>0, \quad h_2(x) = \sup\{x(t)\colon 0\leqslant t\leqslant 1\},$$

$$h_3(x) = \sup\{t\in[0, 1]\colon x(t) = 0\}, \quad h_4(x) = \lambda\{t\in[0, 1]\colon x(t)>0\},$$

$$h_5(x) = \lambda\{t\in[0, h_3(x)]\colon x(t)>0\},$$

where λ denotes the Lebesgue measure.

THEOREM 3. *Under the assumptions of Theorem 2, for every* $1\leqslant i\leqslant 5$, *P-a.s.*

$$D_{N_n}^{-1} \sum_{k=1}^{N_n} d_k \delta h_i^{-1}(Y_k) \Rightarrow Wh_i^{-1} \quad \text{as } n\to\infty, \tag{2.17}$$

where

$$\delta h_1^{-1}(Y_k) = B_k^{-1}\max\{|S_i|^p\colon 0\leqslant i\leqslant k\}, \quad Wh_1^{-1} = \sup\{|W(t)|^p\colon 0\leqslant t\leqslant 1\},$$

$$\delta h_2^{-1}(Y_k) = B_k^{-1}\max\{S_i\colon\ 0 \leqslant i \leqslant k\},$$

$$Wh_2^{-1}((-\infty, x]) = \frac{2}{\sqrt{2\pi}}\int_0^x \exp(-u^2/2)\,du, \qquad x \geqslant 0,$$

$$Wh_3^{-1}((-\infty, x]) = Wh_4^{-1}((-\infty, x]) = \frac{2}{\pi}\arcsin\sqrt{x}, \qquad 0 < x < 1.$$

Let $\{W(t),\ t \geqslant 0\}$ be a one-dimensional Brownian motion starting at 0, on some probability space $(\Omega, \mathscr{A}, P)$, and define the $C[0, 1]$-valued random variables $W^{(s)}$, for $s > 0$, by

(2.18) $$W^{(s)}(u) = s^{-1/2}\,W(su), \qquad u \in [0, 1].$$

THEOREM 4. *Let $k_0 = 1 < k_1 < k_2 < \ldots$ be an increasing sequence of real numbers such that $k_{n+1}/k_n = O(1)$ and $k_n \to \infty$ as $n \to \infty$. Let $\{N_n,\ n \geqslant 1\}$ be a sequence of positive integer-valued random variables such that, for every $n \geqslant 1$, N_n is independent of $\{W(t),\ t \geqslant 0\}$. Put*

(2.19) $$d_n = \log(k_{n+1}/k_n)\exp\left((\log k_n)^{\alpha}\right),$$
$$0 \leqslant \alpha < \tfrac{1}{2},\ D_{N_n} = d_1 + \ldots + d_{N_n},\ n \geqslant 1.$$

If, for some $0 < \varepsilon < \min\left((1-2\alpha)/\alpha, 1\right)$ in the case $0 < \alpha < \frac{1}{2}$, or for some $0 < \varepsilon < 1$ in the case $\alpha = 0$,

(2.20) $$E\left(\log(D_{N_n} \vee 4)\right)^{-(1+\varepsilon)} \to 0 \qquad \text{as } n \to \infty,$$

then the relations

(2.21) $$D_{N_n}^{-1}\sum_{i=1}^{N_n} d_i\,\delta_{W^{(k_i)}} \Rightarrow W, \qquad \text{as } n \to \infty,\ P\text{-a.s.},$$

and

(2.22) $$D_{N_n}^{-1}\sum_{i=1}^{N_n} d_i\,\mu_{W^{(k_i)}} \Rightarrow W, \qquad \text{as } n \to \infty,\ P\text{-a.s.},$$

are equivalent. The result remains valid even in the case when we replace the weight sequence $\{d_n,\ n \geqslant 1\}$, defined by (2.19), *by any sequence $\{d_n^*,\ n \geqslant 1\}$ such that $0 \leqslant d_n^* \leqslant d_n$, $n \geqslant 1$, and $\sum_{n=1}^{\infty} d_n^* = \infty$.*

Let us observe that if, for example, (2.12) holds, then (2.20) and (2.22) also hold. Theorem 4 extends, even in the case $N_n = n$, $n \geqslant 1$, P-a.s., Theorem 1 presented by Rodzik and Rychlik [19] and Proposition 2.1 proved by Fazekas and Rychlik [9].

3. PROOFS

3.1. Proof of Theorem 1. (2.7) $\Rightarrow$ (2.6). Let μ be a given probability distribution. Let us observe that, by Theorem 7.1 of Billingsley [5], Theorem 11.3.3 ($b \Rightarrow c$) of Dudley [8], Lemma 1.4 of Fazekas and Rychlik [9], and Section 2 of Lacey and Philipp [15] (cf. their (6)), it suffices to prove that, for every $f \in BL$,

$$\lim_{n\to\infty} D_{N_n}^{-1} \sum_{k=1}^{N_n} d_k f(\zeta_k) = \int_B f(x)\, d\mu(x) \ \ P\text{-a.s.} \tag{3.1}$$

On the other hand, taking into account (2.7) and Theorem 7.1 of Billingsley [5], we have: For every $f \in BL$,

$$\lim_{n\to\infty} D_{N_n}^{-1} \sum_{i=1}^{N_n} d_k \int_B f(x)\, d\mu_{\zeta_k}(x) \tag{3.2}$$
$$= \lim_{n\to\infty} D_{N_n}^{-1} \sum_{k=1}^{N_n} d_k Ef(\zeta_k) = \int_B f(x)\, d\mu(x) \ \ P\text{-a.s.}$$

Thus, by (3.1) and (3.2), it is enough to prove that for every $f \in BL$

$$\lim_{n\to\infty} D_{N_n}^{-1} \sum_{k=1}^{N_n} d_k \big(f(\zeta_k) - Ef(\zeta_k)\big) = 0 \ \ P\text{-a.s.} \tag{3.3}$$

Let $f \in BL$ be given. Letting now $Z_k = f(\zeta_k) - Ef(\zeta_k)$, we first estimate $|EZ_j Z_k|$ for all $1 \leqslant j \leqslant k < \infty$. We have

$$EZ_k^2 \leqslant Ef^2(\zeta_k) \leqslant (\|f\|_\infty)^2, \quad k \geqslant 1. \tag{3.4}$$

On the other hand, if $1 \leqslant j < k$, then by (2.1) and (2.2) we easily get

$$\begin{aligned} |EZ_j Z_k| &= \big|E\big(f(\zeta_j) - Ef(\zeta_j)\big)\big(f(\zeta_k) - f(\zeta_{j,k}) + f(\zeta_{j,k}) - Ef(\zeta_k)\big)\big| \\ &= \big|E\big(f(\zeta_j) - Ef(\zeta_j)\big)\big(f(\zeta_k) - f(\zeta_{j,k})\big) \\ &\quad + E\big(f(\zeta_j) - Ef(\zeta_j)\big)\big(f(\zeta_{j,k}) - Ef(\zeta_k)\big)\big| \\ &\leqslant E\big|\big(f(\zeta_j) - Ef(\zeta_j)\big)\big|\,\big|\big(f(\zeta_{j,k}) - f(\zeta_k)\big)\big| \leqslant 2\|f\|_\infty E|f(\zeta_{j,k}) - f(\zeta_k)| \\ &\leqslant 2\|f\|_\infty E\{(\|f\|_L \rho(\zeta_{j,k}, \zeta_k)) \wedge (2\|f\|_\infty)\} \leqslant 4\|f\|_{BL}^2 E\{\rho(\zeta_{j,k}, \zeta_k) \wedge 1\} \\ &\leqslant 4\|f\|_{BL}^2 C\{\log_+ \log_+ (c_k/c_j)\}^{-(1+\varepsilon)}. \end{aligned} \tag{3.5}$$

Furthermore, for every j and k, we also have the following inequality:

$$|EZ_j Z_k| \leqslant 4\|f\|_{BL}^2. \tag{3.6}$$

Now, by the independence of the random variable N_n of ζ_k, $k \geqslant 1$, we have

$$E\Big(D_{N_n}^{-1} \sum_{k=1}^{N_n} d_k Z_k\Big)^2 \leqslant 2E\Big(D_{N_n}^{-2} \sum_{j=1}^{N_n} \sum_{k=j}^{N_n} d_k d_j |EZ_j Z_k|\Big). \tag{3.7}$$

Set $\delta_n(j, k) = 1$ if $c_k/c_j \geqslant \exp\big((D_{N_n} \vee 4)^{1/2}\big)$ and $\delta_n(j, k) = 0$ otherwise. Then, by (3.5), we get

$$E\Big(D_{N_n}^{-2} \sum_{j=1}^{N_n} \sum_{k=j}^{N_n} \delta_n(j, k) d_k d_j |EZ_j Z_k|\Big) \tag{3.8}$$

$$\leqslant 2^{3+\varepsilon} C \|f\|_{BL}^2 E\Big(D_{N_n}^{-2} \sum_{j=1}^{N_n} \sum_{k=j}^{N_n} d_j d_k \delta_n(j, k)\big(\log(D_{N_n} \vee 4)\big)^{-(1+\varepsilon)}\Big)$$

$$\leqslant CE\big(\log(D_{N_n} \vee 4)\big)^{-(1+\varepsilon)}.$$

Here, and in what follows, C denotes an absolute constant and the same symbol may be used for different constants.

On the other hand, by (2.3), $M = \sup\{c_{n+1}/c_n \colon n \geqslant 1\} < \infty$. Thus, the relation $c_k/c_j < \exp\big((D_{N_n} \vee 4)^{1/2}\big)$ implies

(3.9) $$\log c_{k+1} - \log c_j = \log(c_{k+1}/c_k) + \log(c_k/c_j) \leqslant \log M + (D_{N_n} \vee 4)^{1/2}.$$

Hence, by (3.6) and (3.9), we obtain

(3.10) $$E\Big(D_{N_n}^{-2} \sum_{j=1}^{N_n} \sum_{k=j}^{N_n} \big(1-\delta_n(j, k)\big) d_j d_k |EZ_j Z_k|\Big)$$

$$\leqslant 4\|f\|_{BL}^2 E\Big(D_{N_n}^{-2} \sum_{j=1}^{N_n} d_j \sum_{k=j}^{N_n} \big(1-\delta_n(j, k)\big)\,(\log c_{k+1} - \log c_k)\Big)$$

$$\leqslant CE\Big(D_{N_n}^{-2} \sum_{j=1}^{N_n} d_j \{\log M + (D_{N_n} \vee 4)^{1/2}\}\Big) \leqslant CE\big(\log(D_{N_n} \vee 4)\big)^{-(1+\varepsilon)}.$$

Using (3.8) and (3.10), we arrive at

(3.11) $$E\Big(D_{N_n}^{-1} \sum_{k=1}^{N_n} d_k Z_k\Big)^2 =: E(T_n)^2 \leqslant CE\big(\log(D_{N_n} \vee 4)\big)^{-(1+\varepsilon)}.$$

Let $\eta > 0$ be so small that $1+\beta = (1+\varepsilon)(1-\eta) > 1$. Let

$$N_{n_k}(\omega) = \min\big\{n = n(\omega) \colon \big(\log(D_{N_n(\omega)} \vee 4)\big)^{-(1+\varepsilon)} \leqslant k^{-(1+\beta)}\big\}$$

$$= \min\{n(\omega) \colon \exp(k^{1-\eta}) \leqslant D_{N_n(\omega)} \vee 4\}$$

and $N_{n_k}(\omega) = \infty$ if, for every $n \geqslant 1$, $D_{N_n}(\omega) \vee 4 < \exp(k^{1-\eta})$. Note that by (2.5) there exists a subsequence $\{N_{n'}, n' \geqslant 1\} \subset \{N_n, n \geqslant 1\}$ such that

$$\big(\log(D_{N_{n'}} \vee 4)\big)^{-(1+\varepsilon)} \to 0, \quad \text{as } n' \to \infty,\ P\text{-a.s.}$$

Thus, $\{N_{n_k}, k \geqslant 1\}$ is a well-defined nondecreasing sequence of positive integer-valued random variables such that

$$E\Big(\sum_{k=1}^{\infty} T_{n_k}^2\Big) = \sum_{k=1}^{\infty} ET_{n_k}^2 \leqslant C \sum_{k=1}^{\infty} E\big(\log(D_{N_{n_k}} \vee 4)\big)^{-(1+\varepsilon)} \leqslant C \sum_{k=1}^{\infty} k^{-(1+\beta)} < \infty.$$

Thus, $\sum_{k=1}^{\infty} T_{n_k}^2 < \infty$, and $\sum_{k=1}^{\infty} \big(\log(D_{N_{n_k}} \vee 4)\big)^{-(1+\varepsilon)} < \infty$ P-a.s. Consequently, $T_{n_k} \to 0$ and $D_{N_{n_k}} \to \infty$, as $k \to \infty$, P-a.s. On the other hand, for $n_k < n \leqslant n_{k+1}$, we have

$$(3.12)\quad |T_n| = D_{N_n}^{-1}\Big|\sum_{i=1}^{N_n} d_i Z_i\Big| \leqslant (D_{N_{n_k}}/D_{N_n})|T_{N_{n_k}}| + 2\|f\|_{BL}^2 D_{N_n}^{-1}\sum_{i=N_{n_k}+1}^{N_n} d_i$$

$$\leqslant |T_{N_{n_k}}| + 2\|f\|_{BL}^2(1 - D_{N_{n_k}}/D_{N_n}).$$

Since, by (2.3) and (2.4), $\sup\{d_n\colon n \geqslant 1\} < \infty$, it follows that P-a.s.

$$(3.13)\quad 1 \leqslant D_{N_n}/D_{N_{n_k}} \leqslant D_{N_{n_{k+1}}}/D_{N_{n_k}}$$

$$= \big(D_{N_{n_{k+1}}-1} + (D_{N_{n_{k+1}}} - D_{N_{n_{k+1}}-1})\big)/D_{N_{n_k}}$$

$$\leqslant \big((\exp(k+1)^{1-\eta}) + \sup\{d_n\colon n \geqslant 1\}\big)/\exp(k^{1-\eta}) \to 1 \quad \text{as } k \to \infty.$$

Hence, by (3.12) and (3.13), we get (2.6).

(2.6) $\Rightarrow$ (2.7). It is sufficient to show that, for every $f \in BL$,

$$(3.14)\quad \lim_{n\to\infty}\Big(D_{N_n}^{-1}\sum_{k=1}^{N_n} d_k \int_B f(x)\,d\mu_{\zeta_k}(x)\Big) = \lim_{n\to\infty}\Big(D_{N_n}^{-1}\sum_{k=1}^{N_n} d_k Ef(\zeta_k)\Big)$$

$$= \int_B f(x)\,d\mu(x) \quad P\text{-a.s.}$$

On the other hand, by (2.6), for every $f \in BL$ we have

$$(3.15)\quad \lim_{n\to\infty}\Big(D_{N_n}^{-1}\sum_{k=1}^{N_n} d_k f(\zeta_k)\Big) = \int_B f(x)\,d\mu(x) \quad P\text{-a.s.}$$

Thus, by (3.14) and (3.15), it remains to prove that

$$(3.16)\quad \lim_{n\to\infty} D_{N_n}^{-1}\sum_{k=1}^{N_n} d_k\big(f(\zeta_k) - Ef(\zeta_k)\big) = 0 \quad P\text{-a.s.}$$

It is easily seen that (3.16) gives (3.3). This completes the proof.

3.2. Proof of Remark 1. In the proof we shall follow some ideas of Berkes and Csáki [3]. Assume that (2.8) holds. Then in Theorem 1 we can choose

$$(3.17)\quad d_k = \log(c_{k+1}/c_k)\exp\big((\log c_k)^\alpha\big), \qquad k \geqslant 1,$$

for some constant $0 \leqslant \alpha < \frac{1}{2}$. Furthermore, in this case, instead of (3.5) we have

$$(3.18)\quad |EZ_jZ_k| \leqslant C(c_j/c_k)^\beta, \qquad 1 \leqslant j < k,\ k \geqslant 2.$$

Let us put $\Delta_n(j, k) = 1$ if $c_k/c_j \geqslant \big(\log(D_{N_n} \vee 4)\big)^{2/\beta}$, and $\Delta_n(j, k) = 0$ otherwise. Then, by (3.18), we get

$$(3.19)\quad E\Big(B_{N_n}^{-2}\sum_{j=1}^{N_n}\sum_{k=j}^{N_n} d_j d_k |EZ_jZ_k|\Delta_n(j, k)\Big)$$

$$\leqslant CE\big(B_{N_n}^{-2}\sum_{j=1}^{N_n}\sum_{k=j}^{N_n} d_j d_k (c_j/c_k)^{\beta} \Delta_n(j,k)\big)$$

$$\leqslant CE\big(B_{N_n}^{-2}\{\log(D_{N_n}\vee 4)\}^{-2}\sum_{j=1}^{N_n}\sum_{k=1}^{N_n} d_j d_k\big) \leqslant CE\big(\log(D_{N_n}\vee 4)\big)^{-2},$$

and, by (3.6),

$$(3.20)\qquad E\big(D_{N_n}^{-2}\sum_{j=1}^{N_n}\sum_{k=j}^{N_n}(1-\Delta_n(j,k))d_j d_k|EZ_jZ_k|\big)$$

$$\leqslant 4\|f\|_{BL}^2 E\big(D_{N_n}^{-2}\sum_{j=1}^{N_n} d_j \sum_{k\in A(j,n)}\{\log c_{k+1}-\log c_k\}\exp((\log c_k)^{\alpha})\big)$$

$$\leqslant CE\big(D_{N_n}^{-2}\exp((\log c_{N_n})^{\alpha})\sum_{j=1}^{N_n} d_j(\log\log(D_{N_n}\vee 4))\big)$$

$$\leqslant CE\big(D_{N_n}^{-1}\{\log\log(D_{N_n}\vee 4)\}\{\exp((\log c_{N_n})^{\alpha})\}\big),$$

where $A(j,n)=\{k\colon c_j\leqslant c_k<c_j(\log(D_{N_n}\vee 4))^{2/\beta}\}$. On the other hand, if (3.17) and (2.3) hold, then P-a.s.

$$D_{N_n}\vee 4 = 4\vee\sum_{k=1}^{N_n} d_k \sim C(\log c_{N_n})^{1-\alpha}\exp((\log c_{N_n})^{\alpha}) \quad \text{as } n\to\infty,$$

and, consequently,

$$\exp((\log c_{N_n})^{\alpha}) \sim CD_{N_n}(\log c_{N_n})^{\alpha-1}, \quad \log c_{N_n}\sim C(\log D_{N_n})^{1/\alpha} \quad \text{as } n\to\infty,$$

$$\exp((\log c_{N_n})^{\alpha}) \sim CD_{N_n}(\log D_{N_n})^{(\alpha-1)/\alpha} \quad \text{as } n\to\infty.$$

Thus, taking into account (3.19) and (3.20), we get

$$(3.21)\qquad E\big(D_{N_n}^{-1}\sum_{k=1}^{N_n} d_k Z_k\big)^2$$

$$\leqslant C\{E\big(\log(D_{N_n}\vee 4)\big)^{-2}+E\big((\log D_{N_n})^{(\alpha-1)/\alpha}\log\log(D_{N_n}\vee 4)\big)\}$$

$$\leqslant CE\big(\log(D_{N_n}\vee 4)\big)^{-(1+\varepsilon)}$$

for every $0<\varepsilon<\min((1-2\alpha)/\alpha,\, 1)$ if $0<\alpha<\frac{1}{2}$; and if $\alpha=0$, then (3.19) and (3.20) give (3.21) for every $0<\varepsilon<1$. Thus, we get (3.11) and the rest of the proof is the same as the arguments in the proof of Theorem 1.

3.3. Proof of Theorem 2. Let $\zeta_n = Y_n$, $n\geqslant 1$, where Y_n is defined by (1.5). Let

$$\zeta_{k,n} = \big(Y_n(t)-S_k/B_n\big)I_{[B_k^2/B_n^2,1)}(t), \qquad t\in[0,1].$$

Then, for every $k<n$, $\zeta_{k,n}$ depends only on $X_{k+1},\ldots,X_n$, and therefore is independent of $Y_k=\zeta_k$. Furthermore, taking into account Doob's inequality,

we also have

$$(3.22)\qquad E\rho(\zeta_{k,n}, \zeta_n) = E\sup\{|\zeta_n(t)-\zeta_{k,n}(t)|:\ 0 \leqslant t \leqslant 1\}$$
$$\leqslant B_n^{-1} E(|S_k|+\max\{|S_j|:\ 1 \leqslant j \leqslant k\})$$
$$\leqslant 2B_n^{-1} E\max\{|S_j|:\ 1 \leqslant j \leqslant k\}$$
$$\leqslant 2B_n^{-1}(E\max\{S_j^2:\ 1 \leqslant j \leqslant k\})^{1/2} \leqslant 2(B_k/B_n).$$

Thus, by (3.22), (2.8) holds with $\beta = \frac{1}{2}$, $c_n = B_n^2$, $n \geqslant 1$. On the other hand, $c_n < c_{n+1}$, $n \geqslant 1$. We also conclude from (1.8) that $c_n \to \infty$, $c_{n+1}/c_n \to 1$ as $n \to \infty$, and finally that (2.3) holds. It is obvious that (2.12) and (2.4) give (2.5) for some $\varepsilon > 0$. Clearly, (1.8) implies (1.9). On the other hand, relations (2.4) and (1.9) imply (2.10). Obviously, by (2.10) and (2.12), we obtain (2.7), which gives (2.6). Thus, Theorem 1 and Remark 1 complete the proof.

3.4. Proof of Theorem 3. In Billingsley [5], cf. Appendix II, it is shown that each of the mappings h_i, $1 \leqslant i \leqslant 5$, is measurable and is continuous except on a set of Wiener measure 0. Therefore Theorem 3 is a consequence of Theorem 2 and (2.16).

3.5. Proof of Theorem 4. Let us put

$$\zeta_n(t) = W^{(k_n)}(t) = \frac{1}{\sqrt{k_n}} W(k_n t), \qquad 0 \leqslant t \leqslant 1,$$

and, for $l < n$,

$$\zeta_{l,n}(t) = \frac{1}{\sqrt{k_n}}\{W(k_n t) - W(k_l)\} I_{(k_l,k_n]}(k_n t), \qquad 0 \leqslant t \leqslant 1.$$

Thus, ζ_l and $\zeta_{l,n}$ are $C[0, 1]$-valued random elements and, for every $l < n$, ζ_l is independent of $\zeta_{l,n}$. On the other hand, by Lemmas 1.11, 1.16 and 1.4 of Freedman [10], we have

$$E\rho(\zeta_n, \zeta_{l,n}) = E\sup\{|\zeta_n(t)-\zeta_{l,n}(t)|:\ 0 \leqslant t \leqslant 1\}$$
$$= \frac{1}{\sqrt{k_n}} E\sup\{|W(k_n t)|:\ 0 \leqslant k_n t \leqslant k_l\}$$
$$= \frac{1}{\sqrt{k_n}} E\sup\{|W(t)|:\ 0 \leqslant t \leqslant k_l\}$$
$$\leqslant \frac{2}{\sqrt{k_n}} E|W(k_l)| \leqslant \frac{2}{\sqrt{k_n}}\{E(W(k_l))^2\}^{1/2} = 2(k_l/k_n)^{1/2}.$$

This gives (2.8) with $\beta = \frac{1}{2}$, $C = 2$ and $c_n = k_n$, $n \geqslant 1$. Clearly, Remark 1 and Theorem 1 complete the proof.

REFERENCES

[1] M. Atlagh, *Théorème central limite presque sûr et loi du logarithme itéré pour des sommes de variables aléatoires indépendantes*, C. R. Acad. Sci. Paris, Sér. I, Probability Theory, 316 (1993), pp. 929–933.

[2] I. Berkes, *Results and problems related to the pointwise central limit theorem*, in: *Asymptotic Methods in Probability and Statistics. A Volume in Honour of Miklós Csörgő*, B. Szyszkowicz (Ed.), Elsevier, Amsterdam 1998, pp. 59–96.

[3] I. Berkes and E. Csáki, *A universal result in almost sure central limit theory*, Stochastic Process. Appl. 94 (2001), pp. 105–134.

[4] I. Berkes, E. Csáki and S. Csörgő, *Almost sure limit theorems for the St. Petersburg game*, Statist. Probab. Lett. 45 (1999), pp. 23–30.

[5] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York 1968.

[6] G. A. Brosamler, *An almost everywhere central limit theorem*, Math. Proc. Cambridge Philos. Soc. 104 (1988), pp. 561–574.

[7] M. Csörgő and L. Horváth, *Invariance principles for logarithmic averages*, Math. Proc. Cambridge Philos. Soc. 112 (1992), pp. 195–205.

[8] R. M. Dudley, *Real Analysis and Probability*, Wadsworth, Belmont, CA, 1989.

[9] I. Fazekas and Z. Rychlik, *Almost sure functional limit theorems*, Ann. Univ. Mariae Curie-Skłodowska, Lublin–Polonia, Sectio A, Vol. LVI, 1 (2002), pp. 1–18.

[10] D. Freedman, *Brownian Motion and Diffusion*, Holden-Day, 1971.

[11] A. Gut, *Stopped Random Walks: Limit Theorems and Applications*, Springer, New York 1988.

[12] I. A. Ibragimov, *On almost sure versions of limit theorems* (in Russian), Dokl. Akad. Nauk 350 (1996), pp. 301–303.

[13] I. A. Ibragimov and M. A. Lifshits, *On the convergence of generalized moments in almost sure central limit theorem*, Statist. Probab. Lett. 40 (1998), pp. 343–351.

[14] I. A. Ibragimov and M. A. Lifshits, *On almost sure limit theorems*, Theory Probab. Appl. 44 (1999), pp. 254–272.

[15] M. T. Lacey and W. Philipp, *A note on the almost sure central limit theorem*, Statist. Probab. Lett. 9 (1990), pp. 201–205.

[16] P. Major, *Almost sure functional limit theorems. Part I. The general case*, Studia Sci. Math. Hungar. 34 (1998), pp. 273–304.

[17] P. Major, *Almost sure functional limit theorems. Part. II. The case of independent random variables*, Studia Sci. Math. Hungar. 36 (2000), pp. 231–273.

[18] Yu. V. Prokhorov, *Convergence of random processes and limit theorems in probability theory*, Teor. Veroyatnost. i Primenen. 1 (1956), pp. 177–238; English translation in: Theory Probab. Appl. 1 (1956), pp. 157–214.

[19] B. Rodzik and Z. Rychlik, *An almost sure central limit theorem for independent random variables*, Ann. Inst. H. Poincaré 30 (1994), pp. 1–11.

[20] Z. Rychlik and K. S. Szuster, *Some remarks on the almost sure central limit theorem for independent random variables*, Probab. Math. Statist. 23 (2003), pp. 241–249.

[21] Z. Rychlik and D. Szynal, *A functional random-sum central limit theorem*, Bull. Acad. Polon. Sci., Sér. Math. Astronom. Phys. 23 (1975), pp. 1013–1018.

[22] P. Schatte, *On strong versions of the central limit theorem*, Math. Nachr. 137 (1988), pp. 249–256.

[23] P. Schatte, *On the central limit theorem with almost sure convergence*, Probab. Math. Statist. 11 (1991), pp. 237–246.

[24] P. Schatte, *Two remarks on the almost sure central limit theorem*, Math. Nachr. 154 (1991), pp. 225–229.

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